

## ASYMPTOTIC BEHAVIOUR OF NON-PARAMETRIC MINIMAL HYPERSURFACES

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### 1. Introduction

The main result of this paper is Theorem II, which deals with the asymptotic behaviour of non-parametric complete minimal hypersurfaces. In [2], Bombieri, De Giorgi and Giusti showed that hypersurfaces of this type other than hyperplanes exist for dimensions greater than seven. For lower dimensions Theorem II is vacuous because, by theorems of Almgren [1], De Giorgi [6], and Simons [7], the only hypersurfaces of this type are the hyperplanes. The proof of Theorem II relies on Theorem I, which states that the Gauss map of a hypersurface with constant mean curvature is harmonic. A map of Riemannian manifolds is called harmonic if it is an extremal of a certain energy integral which generalizes the classical Dirichlet integral. An extensive study of harmonic maps has been done by Eells and Sampson [3]. If the dimension of a minimal surface is two, then the Gauss map is holomorphic; this is much stronger than harmonic. We expect that the weaker property will still be useful in extending to higher dimensions some of the theorems on 2-dimensional minimal surfaces which are obtained through complex function theory. An example of this method is the proof of Theorem II.

### 2. Harmonic maps

At this point we begin the discussion of Theorem I. A map  $f: M_1 \rightarrow M_2$  of Riemannian manifolds is called harmonic if  $f$  is an extremal of the integral

$$E(f) = \int \text{Tr } f^*g \, dv ,$$

where  $\text{Tr } f^*g$  denotes the trace of the pullback under  $f$  of the Riemannian metric  $g$  on  $M_2$ , and  $dv$  denotes the volume form on  $M_1$ . For our purposes, the map in question will be the Gauss map  $n: M \rightarrow S^n$  which sends a point on  $M$  into its unit normal vector. Now we are in a position to state the first theorem.

**Theorem I.** *If  $i: M \rightarrow E^{n+1}$  is an isometric immersion of the  $n$ -dimensional*

manifold  $M$  as a hypersurface with constant mean curvature, then the Gauss map  $n: M \rightarrow S^n$  is harmonic.

**Note.** The mean curvature at a point is the sum of the principal curvatures. In the case of a minimal hypersurface, the mean curvature is zero.

*Proof.* If  $f = n + \varepsilon h$  is a variation of  $n: M \rightarrow S^n \subset E^{n+1}$ , then the integrand,  $Tr f^*g$ , which occurs in the definition of a harmonic mapping can be written as  $Tr \langle \nabla(n + \varepsilon h), \nabla(n + \varepsilon h) \rangle$ , where  $\langle , \rangle$  and  $\nabla$  denote inner product and covariant derivative respectively in  $E^{n+1}$  or their restrictions to  $M$ . The vector  $h$ , for a given point  $x \in M$ , is in the tangent space of  $S^n$  at  $n(x)$ . Since the tangent spaces of  $M$  and  $S^n$  are parallel,  $h$  will be identified with a vector of the same name in the tangent space of  $M$ . If we denote the derivative at  $\varepsilon = 0$  of the energy  $E(f)$  by  $E'(n)$ , then we have

$$E'(n) = \int 2 Tr \langle \nabla n, \nabla h \rangle dv ,$$

where  $dv$  denotes the volume form on  $M$ . We will show that the above integral is zero for a variation which leaves the boundary fixed. In fact, we will show

$$(1) \quad E'(n) = 2 \int d\star \langle \nabla n, h \rangle ,$$

where  $d$  denotes the exterior derivative, and  $\star$  the star operator on exterior forms on  $M$ .

An application of Stokes' theorem shows that the right hand side of (1) is zero for variations with fixed boundary because  $h = 0$  on the boundary.

Now we proceed to prove (1). Since the covariant derivative of  $\langle , \rangle$  is zero, we obtain  $\langle \nabla n, \nabla h \rangle = \nabla \langle \nabla n, h \rangle - \langle \nabla \nabla n, h \rangle$ . First we observe that  $Tr \nabla \langle \nabla n, h \rangle dv = d\star \langle \nabla n, h \rangle$ . Second,  $Tr \langle \nabla \nabla n, h \rangle$  is shown to be zero as follows: The Codazzi-Mainardi equations, together with the fact that  $\langle \nabla n, \rangle$  is symmetric, imply that  $\langle \nabla \nabla n, \rangle$  is symmetric in all three arguments. For a proof see Kobayashi and Nomizu [7, Corollary 4.4, p. 25]. Using this property we obtain

$$Tr \langle \nabla \nabla n, h \rangle = \nabla_h Tr \langle \nabla n, \rangle = \nabla_h H = 0 ,$$

where  $H$  denotes the mean curvature which is constant by assumption. This completes the proof of Theorem I.

### 3. Main theorem

For Theorem II, which is the main theorem of this paper, the manifold  $M$  is a non-parametric hypersurface, i.e., it is the graph of a real function of  $n$  variables. We denote the angle of the normal vector to the manifold with the  $(n + 1)$ -st coordinate axis by  $\varphi$ . If the manifold is properly oriented, then

$0 \leq \varphi < \pi/2$ . We denote by  $M(t)$  the intersection of the image of  $M$  under the map of  $E^{n+1}$  into itself defined by  $x \rightarrow (1/t)x$  with the unit ball centered at the origin in  $E^{n+1}$ . Now we state the main theorem.

**Theorem II.** *If  $M$  is a complete non-parametric minimal hypersurface, then  $M$  is either a hyperplane, or the limit of the average of  $\varphi$  over  $M(t)$  as  $t$  tends to infinity is equal to  $\pi/2$ .*

For the proof of Theorem II, the concept of integral currents, introduced by Federer and Fleming, will be adopted. Although the powerful theorems of this theory are only used in a very weak form, this concept proves to be convenient. Integral currents are the natural “domains of integration” for the integral theorems we plan to use.

**Note.** Readers not familiar with currents may substitute manifolds. This does not change the essential arguments.

The proof of Theorem II is divided into three steps. In step one, formula (1) in the proof of Theorem I is applied to a distance increasing variation of the Gauss map. In step two, this formula is used to obtain an estimate for a certain curvature measure of  $M(t)$  for large  $t$ . In step three, the estimate obtained in step two is used to prove the alternative asserted in Theorem II.

*Step 1.* The variation of the Gauss map mentioned above will be defined in terms of a canonical coordinate system  $\exp: B \rightarrow H \subset S^n$ , where  $B$  is a ball of radius  $\pi/2$  in  $E^n$  and  $H$  is a hemisphere containing the image of the Gauss map. The map  $\exp$  sends lines through the center of  $B$  into great circles through the pole of  $H$ . The variation of the Gauss map  $n$  in terms of this coordinate system is described by the map  $B \rightarrow B$  which sends  $y$  into  $(1 + \epsilon)y$ . This variation is distance increasing on  $H$ . Consequently, the rate of increase of the energy integral can be given by  $\int K dv$ , where the integrand  $K$  is non-negative on  $M(t)$ . The properties of  $K$  will be further discussed in step three.

According to Theorem I, the Gauss map  $n$  is harmonic, and we can apply equation (1) in the proof of Theorem I. For the variation defined above, the term  $2\langle \nabla n, h \rangle$  is equal to  $d\varphi^2$ , where  $\varphi$  is the angle introduced in Theorem II and  $d$  is the exterior derivative. Therefore we obtain

$$\int K dv = \int d^*d \varphi^2 .$$

If  $T(t)$  denotes the current defined by integration of  $n$ -forms in  $E^{n+1}$  over  $M(t)$ , then the above formula can be written as follows:

$$(2) \quad \int K dv = T(t)(d^*d \varphi^2) = \partial T(t)(*d\varphi^2) ,$$

where  $\partial$  denotes the boundary operator and the form  $*d\varphi^2$ , originally defined on  $M(t)$ , has been extended arbitrarily to a form on  $E^{n+1}$  for which the same symbol is used.

*Step 2.* To obtain the curvature estimate mentioned in the outline, the derivative  $\Phi'(t)$  of the function  $\Phi(t) = \int \varphi^2 dv$  will be estimated. The integration is over  $M(t)$  and  $\varphi$  is the angle introduced earlier. The manifold  $M(t + \Delta t)$  can be interpreted as a normal variation of  $M(t)$ . Since  $M(t)$  is minimal, the rate of change in area will consist of a boundary integral alone. This boundary integral is positive because, as shown in [5], the area of  $M(t)$  is non-decreasing. This simplifies the computation of  $\Phi'(t)$ . In fact,  $\Phi'(t)$  can be expressed as the sum of the integral  $(1/t) \langle \text{grad } \varphi^2, x \rangle$  over  $M(t)$ , where  $x \in E^{n+1}$  denotes the coordinate of a point  $p \in M(t)$ , and a boundary integral which is positive because the area of  $M(t)$  is increasing. The first summand reflects the fact that the first order displacement of a point is given by  $(\Delta t/t)x$ . Instead of  $\langle \text{grad } \varphi^2, x \rangle dv$  on  $M$  we write  $r * d\varphi^2 \wedge dr$  where  $r = |x|$ , and obtain the following formula:

$$(3) \quad \Phi'(t) \geq \frac{1}{t} \int r * d\varphi^2 \wedge dr = \frac{1}{t} T(t)(r * d\varphi^2 \wedge dr).$$

Using a formula which deals with slicing of currents, we will replace the right hand side of equation (3). The formula is trivial if the slices are represented by smooth manifolds but this may not be the case here. In the following,  $T(t, r)$  will denote the restriction of  $T(t)$  to a ball of radius  $r$  centered at the origin. Equation (3) of Corollary 3.6 in Federer [4] yields

$$(4) \quad T(t)(r * d\varphi^2 \wedge dr) = \int_0^1 [\partial T(t, r)(r * d\varphi^2)] dr.$$

Combining formulas (2), (3), and (4) we obtain

$$(5) \quad \Phi'(t) \geq \frac{1}{t} \int_0^1 r \int_{M(t, r)} K dv dr,$$

where  $M(t, r)$  is the restriction of  $M(t)$  to a ball of radius  $r$ .

Since  $\Phi(t)$  is bounded and  $\int K dv$  is continuous in  $1/t$ , equation (5) implies that the limit of  $\int K dv$ , as  $t$  tends to infinity, is zero. This will be used in step three.

*Step 3.* The variation introduced in step 1 was chosen because the rate of increase of the energy integral denoted by  $\int K dv$  can be estimated easily. In fact,  $K \geq \|\text{grad } \varphi^2\|^2$  holds as well as  $K \geq \alpha(\varphi)K_2$ , where  $K_2$  denotes the sum of squares of the principal curvatures and  $\alpha(\varphi)$  is equal to  $\sin \varphi \cos \varphi$ . The first inequality and the result of step 2 imply that the integral

$\int \|\text{grad } \varphi\|^2 dv$  tends to zero as  $t$  tends to infinity. This in turn implies that there exists a number  $\varphi_0 = \varphi_0(t)$  such that  $\int |\varphi - \varphi_0| dv$  is arbitrarily small for large  $t$ . The reference is to Morrey [8, Theorem 3.6·5]. In other words,  $\varphi$  is arbitrarily close to  $\varphi_0$  except possibly for a set of small measure on  $M(t)$ . To complete the proof of Theorem II we show that either  $\varphi_0$  is in any neighborhood of  $\pi/2$  for large enough  $t$  or  $M$  is a hyperplane. If  $\lim \varphi_0(t) \neq \pi/2$  for some sequence  $t_n \rightarrow \infty$ , then the second inequality  $K \geq \alpha(\varphi)K_2$  implies either  $\int K_2 dv \rightarrow 0$ , or  $\lim \varphi_0 = 0$ .

One way to conclude the proof is with a second application of the argument involving the reference to Morrey [8] which shows that the image of the Gauss map is arbitrarily close to a point  $n_0 \in S^n$  except possibly for a set of small measure. This then implies that  $M$  is a hyperplane. A shorter proof can be obtained by using the fact that a subsequence of  $T(t)$  converges to a cone, see Fleming [5]. The above conditions on  $M(t)$  then imply that the cone is a hyperplane.

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